THE SYMPLECTIC FLOER HOMOLOGY OF THE FIGURE EIGHT KNOT

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Abstract. In this paper, we compute the symplectic Floer homology of the figure eight knot. This provides first nontrivial knot with trivial symplectic Floer homology.

1. Introduction

In [3], we generalized the Casson-Lin invariant [5] to the symplectic theory point of view. Our symplectic Floer homology of knots serves a new invariant for knots, and its Euler characteristic is half of the signature of knots.

We showed that the symplectic Floer homology of the unknotted knot is trivial in [3]. The natural question arises as whether there is a nontrivial knot with trivial symplectic Floer homology. We answer this question in this paper by computing the symplectic Floer homology of the figure eight knot.

Although we know that the signature of the figure eight knot $4_1 = \overline{\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}}$ is zero, the signature does not suffice to give the information of our finer invariant - the symplectic Floer homology. For the square knot, we computed in [4] that the symplectic Floer homology is nontrivial even though its signature is zero. Our main result is the following.

Theorem *The symplectic Floer homology of the figure eight knot* $4_1 = \overline{\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}}$ *is*

$$HF_i^{sym}(4_1) = CF_i^{sym}(4_1) = 0, \quad \textit{for all } i \in \mathbf{Z}_4.$$

To our knowledge, this is the first trivial symplectic Floer homology involving nontrivial information. It is still an open question about if there is a non-homotopy 3-sphere with trivial instanton Floer homology. We wish to build the relation between our symplectic Floer homology of knots [3] and the instanton Floer homology of homology 3-spheres [1] through the Dehn surgery technique. Using the calculation of the figure eight knot, we hope to find an example of non-homotopy 3-sphere with trivial instanton Floer homology.

2. The symplectic Floer homology

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2.1. **The symplectic Floer homology of braids.** We briefly recall our definition of the Floer homology of braids in this subsection. See [3] for more details.

For any knot $K = \overline{\beta}$ with $\beta \in B_n$, the braid group, the space $\mathcal{R}(S^2 \setminus K)^{[i]}$ can be identified with the space of $2n$ matrices $X_1 \cdots, X_n$, $Y_1, \cdots, Y_n$ in $SU(2)$ satisfying

$$\operatorname{tr}(X_i) = \operatorname{tr}(Y_i) = 0, \quad \text{for } i = 1, \cdots, n, \tag{1}$$

$$X_1 \cdot X_2 \cdots X_n = Y_1 \cdot Y_2 \cdots Y_n. \tag{2}$$

Note that $\pi_1(S^2 \setminus K)$ is generated by $m_{x_i}, m_{y_i} (i = 1, 2, \cdots, n)$ with one relation $\prod_{i=1}^n m_{x_i} = \prod_{i=1}^n m_{y_i}$. There is a unique reducible conjugacy class of representations $s_K : \pi_1(S^3 \setminus K) \to U(1)$ such that

$$s_K([m_{x_i}]) = s_K([m_{y_i}]) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Let $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ be the subset of $\mathcal{R}(S^2 \setminus K)^{[i]}$ consisting of irreducible representations. Then $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ is a monotone symplectic manifold of dimension $4n - 6$ by Lemma 2.3 in [3]. The symplectic manifold (M, ω) is called *monotone* if $\pi_2(M) = 0$ or if there exists a nonnegative $\alpha \geq 0$ such that $I_\omega = \alpha I_{c_1}$ on $\pi_2(M)$, where $I_\omega(u) = \int_{S^2} u^*(\omega) \in \mathbf{R}$ and $I_{c_1}(u) = \int_{S^2} u^*(c_1) \in \mathbf{Z}$ for $u \in \pi_2(M)$. The braid β induces a diffeomorphism $\phi_\beta : \mathcal{R}^*(S^2 \setminus K)^{[i]} \to \mathcal{R}^*(S^2 \setminus K)^{[i]}$. The induced diffeomorphism ϕ_β is symplectic, and the fixed point set of ϕ_β is $\mathcal{R}^*(S^3 \setminus K)^{[i]}$ (see Lemma 2.4 in [3]).

Let $H : \mathcal{R}^*(S^2 \setminus K)^{[i]} \times \mathbf{R} \to \mathbf{R}$ be a C^∞ time-dependent Hamiltonian function with $H(x, s) = H(\phi_\beta(x), s + 1)$. Let X_s be the corresponding vector field from $\omega(X_s, \cdot) = dH_s(\cdot, s)$, and ψ_s be the corresponding flow

$$\frac{d\psi_s}{ds} = X_s \circ \psi_s, \quad \psi_0 = id.$$

Then we have $\psi_{s+1} \circ \phi_\beta^H = \phi_\beta \circ \psi_s$, where $\phi_\beta^H = \psi_1^{-1} \circ \phi_\beta$. Let Ω_{ϕ_β} be the space of smooth paths α in $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ such that $\alpha(s + 1) = \phi_\beta(\alpha(s))$. The symplectic action $a_H : \Omega_{\phi_\beta} \to \mathbf{R}/2\alpha N\mathbf{Z}$ is given by

$$da_H(\gamma)\xi = \int_0^1 \omega(\dot{\gamma} - X_s(\gamma), \xi) ds.$$

So the critical points of a_H are the fixed points of ϕ_β^H. For $x \in \operatorname{Fix}(\phi_\beta^H)$, define $\mu(x) = \mu_u(x, s) \pmod{2N}$, where μ_u is the Maslov index and $N = N(K)$ is the minimal value of the first Chern number of the tangent bundle of $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. The integer $N(K)$ is a knot invariant.

Thus we have a $\mathbf{Z}_{2N}$-graded symplectic Floer chain complex:

$$CF_i^{\text{sym}} = \{x \in \operatorname{Fix}(\phi_\beta) \cap \mathcal{R}^*(S^2 \setminus K)^{[i]} : \mu(x) = i\}, \quad i \in \mathbf{Z}_{2N}.$$

The following is Proposition 4.1 and Theorem 4.2 of [3].

Theorem 2.1. *For a knot $K = \overline{\beta}$ with the property that $\pi_2(\mathcal{R}^*(S^2 \setminus K)^{[i]}) = 0$ or $\alpha N(K) = 0$, there is a well-defined $\mathbf{Z}$-graded symplectic Floer homology $HF_*^{sym}(\phi_\beta)$. The symplectic Floer homology $\{HF_i^{sym}(\phi_\beta)\}_{i \in \mathbf{Z}_{2N}}$ is a knot invariant and its Euler number is half of the signature of the knot (see* [3]*).*

2.2. **The symplectic Floer homology of the figure eight knots.** The figure eight knot 4_1 has the braid representative $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$. The knot 4_1 has signature zero since 4_1 is equivalent (by an orientation preserving homeomorphism) to its mirror image $\overline{4_1}$. So the figure eight knot is amphicheiral. Also it is well-known that the figure eight knot is not a slice knot, and represents an element of order 2 in the knot cobordism group (see [6]).

We calculate the symplectic Floer homology of the figure eight knot by identifying the fixed points of the induced symplectic diffeomorphism in §2.1.

Let $\mathcal{R}^*(S^2 \setminus 4_1)^{[i]}$ be the subset of $\mathcal{R}(S^2 \setminus 4_1)^{[i]}$ consisting of irreducible representations. Then $\mathcal{R}^*(S^2 \setminus 4_1)^{[i]}$ can be also identified with $(H_3 \setminus S_3)/SU(2)$ in Lin's notation [5], i.e., the set of 6-tuple $(X_1, X_2, X_3, Y_1, Y_2, Y_3) \in SU(2)^6$ satisfying $\mathrm{tr}(X_j) = \mathrm{tr}(Y_j) = 0 (j = 1, 2, 3)$ and

$$X_1X_2X_3 = Y_1Y_2Y_3.$$

By operating the conjugation on X_3 and Y_3, we may assume that

$$X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Y_3 = \begin{pmatrix} i\cos\theta & \sin\theta \\ -\sin\theta & -i\cos\theta \end{pmatrix}, \quad 0 \le \theta \le \pi.$$

If $\theta = 0$ and π, then we get two copies of $(H_2 \setminus S_2)/SU(2)$ which is the pillow case (a 2-sphere with four cone points deleted [3, 5]). For $0 < \theta < \pi$, the identification reduces down to the following

$$X_1X_2 \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} = Y_1Y_2.$$

Let R_θ be the representations in $\mathcal{R}^*(S^2 \setminus 4_1)^{[i]}$ satisfying the above equation. So the space R_θ is the non-singular piece in $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. For $0 < \theta, \theta' < \pi$, the space R_θ is diffeomorphic to the space $R_{\theta'}$. In particular, they are all diffeomorphic to $R_{\pi/2}$. In this case, we see that $\mathcal{R}^*(S^2 \setminus 4_1)^{[i]}$ is a generalized pillow case:

$$\mathcal{R}^*(S^2 \setminus 4_1)^{[i]} = \bigcup_{0 \le \theta \le \pi} R_\theta.$$

The fixed point set of ϕ_{4_1} is $\mathcal{R}^*(S^3 \setminus 4_1)^{[i]}$ by Lemma 2.4 in [3]. So we have, for $\sigma = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, $\mathrm{Fix}\,(\phi_{4_1}) = \{(X_1, X_2, X_3) \in SU(2)^3 | \sigma(X_j) = X_j, j = 1, 2, 3\}$ up to conjugation. Let B_n be the braid group of rank n with the standard generators $\sigma_1, \cdots, \sigma_{n-1}$, and F_n be the free group of rank n generated by $x_1, \cdots, x_n$. Then the automorphism of F_n

representing σ_k is given by (still denote it by σ_k)

$$\begin{aligned} \sigma_k: \quad & x_k \mapsto x_k x_{k+1} x_k^{-1} \\ & x_{k+1} \mapsto x_k \\ & x_l \mapsto x_l, \quad l \neq k, k+1. \end{aligned} \tag{3}$$

By (3), we compute the followings.

$$\begin{aligned} \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(x_1) &= \sigma_1\sigma_2^{-1}\sigma_1(x_1^{-1}) = \sigma_1\sigma_2^{-1}(x_1x_2^{-1}x_1^{-1}) \\ &= \sigma_1(x_1x_2x_3x_2^{-1}x_1^{-1}) \\ &= (x_1x_2x_1^{-1})x_1x_3x_1^{-1}(x_1x_2x_1^{-1})^{-1} \\ &= x_1x_2x_3x_2^{-1}x_1^{-1}. \\ \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(x_2) &= \sigma_1\sigma_2^{-1}\sigma_1(x_2x_3^{-1}x_2^{-1}) \\ &= \sigma_1\sigma_2^{-1}(x_1x_3^{-1}x_1^{-1}) = \sigma_1(x_1^{-1}x_2x_1) \\ &= x_1x_2^{-1}x_1x_2x_1^{-1}. \\ \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(x_3) &= \sigma_1\sigma_2^{-1}\sigma_1(x_2^{-1}) = \sigma_1\sigma_2^{-1}(x_1^{-1}) \\ &= \sigma_1(x_1) = x_1x_2x_1^{-1}. \end{aligned}$$

Therefore the fixed point set of ϕ_{4_1} is the set of points $(X_1, X_2, X_3) \in SU(2)^3$ such that

$$\begin{aligned} \operatorname{tr}(X_j) = 0, \quad j = 1, 2, 3, \\ X_1X_2X_3X_2^{-1}X_1^{-1} &= X_1, \\ X_1X_2^{-1}X_1X_2X_1^{-1} &= X_2, \\ X_1X_2X_1^{-1} &= X_3, \end{aligned}$$

up to conjugation. Up to conjugation, we can assume that

$$X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_1 = \begin{pmatrix} i\cos\theta & \sin\theta \\ -\sin\theta & -i\cos\theta \end{pmatrix}, \quad 0 \le \theta \le \pi.$$

From the last equation in the above, we obtain

$$\begin{aligned} X_1X_2X_1^{-1} &= \begin{pmatrix} i\cos\theta & \sin\theta \\ -\sin\theta & -i\cos\theta \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} -i\cos\theta & -\sin\theta \\ \sin\theta & i\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} i\cos 2\theta & \sin 2\theta \\ -\sin 2\theta & -i\cos 2\theta \end{pmatrix} = X_3. \end{aligned}$$

So the matrix X_3 is completely determined by the parameter $\theta \in [0, \pi]$. This is, in fact, a key to complete the calculation. Now substituting X_3 into the relation $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(X_1) = X_1$, we have

$$\begin{aligned} X_1X_2X_3^{-1}X_2^{-1}X_1^{-1} &= \begin{pmatrix} -\cos\theta & -i\sin\theta \\ -i\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} -i\cos 2\theta & \sin 2\theta \\ -\sin 2\theta & i\cos 2\theta \end{pmatrix} \begin{pmatrix} -\cos\theta & i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -i\cos 4\theta & -\sin 4\theta \\ \sin 4\theta & i\cos 4\theta \end{pmatrix} = X_1. \end{aligned}$$

This reduces to the equations

$$\cos 4\theta = -\cos\theta, \qquad \sin 4\theta = -\sin\theta. \tag{4}$$

Similarly, we compute

$$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}(X_2) = \begin{pmatrix} i\cos 3\theta & \sin 3\theta \\ -\sin 3\theta & -i\cos 3\theta \end{pmatrix} = X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

to get the equations

$$\cos 3\theta = 1, \qquad \sin 3\theta = 0. \tag{5}$$

Thus the fixed point of ϕ_{4_1} can be identified with

$$X_1 = \begin{pmatrix} i\cos\theta & \sin\theta \\ -\sin\theta & -i\cos\theta \end{pmatrix}, X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_3 = \begin{pmatrix} i\cos 2\theta & \sin 2\theta \\ -\sin 2\theta & -i\cos 2\theta \end{pmatrix}, \quad 0 \le \theta \le \pi,$$

subject to equations (4) and (5). Using the equations (5) and the angle addition formulae for sine and cosine functions with $4\theta = 3\theta + \theta$, (4) becomes

$$\sin\theta = 0, \quad \cos\theta = 0. \tag{6}$$

There is no solution for (6). Hence

$$\mathrm{Fix}\,(\phi_{4_1}) = \emptyset \quad \text{(empty set)}. \tag{7}$$

Theorem 2.2. *The symplectic Floer homology of the figure eight knot* $4_1 = \overline{\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}}$ *is*

$$HF_i^{sym}(4_1) = CF_i^{sym}(4_1) = 0, \quad \textit{for all } i \in \mathbf{Z}_{2N}.$$

Proof: Since the $\mathbf{Z}_{2N}$-graded symplectic Floer chain complex $CF_i^{\mathrm{sym}}(4_1)$ is generated by $\mathrm{Fix}\,(\phi_{4_1})$, the result follows from (7). □

2.3. **The symplectic Floer homology of knots with braid representatives in B_3.** It seems that the method in §2.2 can be adapted to knots with braid representatives in B_3. We are going to illustrate another example to show that the computation for the figure eight knot in §2.2 is quite lucky.

Let $K = 5_2$ be the knot with 5-crossings. We have the braid representative $\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_2$ for the knot 5_2 (see [6]). Thus the fixed points of ϕ_{5_2} can be identified, by the same method in §2.2, with the set of points $(X_1, X_2, X_3) \in SU(2)^3$ such that

$$\begin{aligned} \mathrm{tr}(X_j) &= 0, \quad j = 1,2,3, \\ X_1X_2X_3X_1X_2^{-1}X_1^{-1}X_3^{-1}X_2^{-1}X_1^{-1} &= X_1, \\ X_1X_2X_3^{-1}X_1^2X_2^{-1}X_1^{-1} &= X_2, \\ X_1X_2X_1^{-1}X_2^{-1}X_1^{-1} &= X_3, \end{aligned}$$

up to conjugation. This follows a straightforward calculation of $\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_2(x_j)(j = 1,2,3)$. Again we can compute X_3 from the last equation in the above.

$$X_1X_2X_1^{-1}X_2^{-1}X_1^{-1} = \begin{pmatrix} -i\cos 3\theta & -\sin 3\theta \\ \sin 3\theta & i\cos 3\theta \end{pmatrix} = X_3.$$

Then $\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_2(X_j) = X_j (j = 1, 2)$ gives us

$$\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_2(X_1) = \begin{pmatrix} -i\cos 6\theta & -\sin 6\theta \\ \sin 6\theta & i\cos 6\theta \end{pmatrix} = X_1$$
$$\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_2(X_2) = \begin{pmatrix} -i\cos 5\theta & -\sin 5\theta \\ \sin 5\theta & i\cos 5\theta \end{pmatrix} = X_2.$$

Thus we need to solve the equations

$$\cos 6\theta = -\cos\theta, \quad \sin 6\theta = -\sin\theta, \cos 5\theta = -1, \quad \sin 5\theta = 0. \tag{8}$$

There are three solutions of (8) with $\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi$. Let $\rho_j (j = 1, 2, 3)$ be the corresponding fixed points of ϕ_{5_2} in $\mathcal{R}^*(S^2 \setminus 5_2)^{[i]}$.

By following the method in [2], for $K = 5_2$, we have all type I double points so that the correction term $\mu = 0$. Using the definition of Goeritz matrix in §1 of [2], we get the Goeritz matrix of 5_2:

$$G(5_2) = \begin{pmatrix} 4 & -3 & -1 \\ -3 & 4 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

By the theorem 6 of [2], we have

$$\text{Signature}(5_2) = \text{Signature}(G(5_2)) - \mu = 2.$$

By Theorem 2.1, the Euler characteristic of the symplectic Floer homology of 5_2 is one.

Proposition 2.3. *The symplectic Floer chain complex of 5_2 is given by: one of the odd chain groups is generated by one of $\rho_j (j = 1, 2, 3)$; even chain groups are generated by the rest two fixed points of ϕ_{5_2}.*

It is nontrivial to determine the Maslov index of ρ_j and the possible Floer boundary map in order to complete the calculation.

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